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Dynamical partitions of space in any dimension

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Abstract. Topologically stable cellular partitions of D-dimensional spaces are studied. A complete statistical description of the average structural properties of such partitions is given in terms of a sequence of $\frac{D}{2} - 1$ (or $\frac{D-1}{2}$) variables for D even (or odd). These variables are the average coordination numbers of the $2k$-dimensional polytopes ($2k < D$) which make up the cellular structure. A procedure to produce D-dimensional space partitions through cell-division and cell-coalescence transformations is presented. Classes of structures which are invariant under these transformations are found and the average properties of such structures are illustrated. Homogeneous partitions are constructed and compared with the known structures obtained by Voronoï partitions and sphere packings in high dimensions.

1. Introduction

We study the topologically stable division of any dimensional space by cells. Such systems have minimal incidence numbers. Configurations with higher incidence numbers are topologically unstable because they can be split into configurations with the minimal incidence numbers by infinitesimal local transformations. In the literature these cellular partitions are known as 'froths' since in two and three dimensions (2D and 3D) a soap froth is the archetype of such structures. A 2D froth is a space-filling cellular partition made of irregular polygons where three polygons are incident on each vertex. A 3D froth is a polyhedral partition of space where on each vertex are incident four polyhedra. In general, a D-dimensional froth is a partition of space in irregular polytopes, where on each vertex $D + 1$ polytopes are incident. Cellular structures with minimal incidence numbers always appear when the space is filled by cells without following any special symmetry. Therefore, froths are the typical structures of any disordered partition of space into cells.

A broad class of disordered natural and artificial cellular systems have the topological structure of froths [1–3]. Examples in 2D are magnetic domains in garnet films, Bérnard–Marangoni cells in thermal convections, biological tissues, cuts of polycrystalline metals and ceramics, emulsions, the subdivision of territory into administrative regions or states, geological structures and 2D soap froth (which is obtained by squeezing a soap foam between two plates) [4–6]. In 3D, examples are: biological cells, polycrystalline metals and ceramics, and foams [4, 7, 8]. Moreover, the structure of any packing (of hard spheres or atoms, for example) is the dual of a cellular system (which can be generated, for instance, by using the Voronoï construction [9] around the centres of the packed elements). In general, cellular systems generated by packing elements without the use of any specific symmetry

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have structures which are topologically froths. It follows that, among the examples of 3D froths one can include amorphous metals, glasses and some crystalline structures such as the tetrahedrally close-packed phases [10, 11].

Froths in spaces of dimensionality higher than three are relevant in information theory and signal processing [12, 13]. Indeed, an information can be associated with a point in an N-dimensional space. To transmit and recover the information in the presence of noise one must put the points in the N-dimensional space, separated by a certain distance which must be larger than the additional noise. Therefore to each point (information) is associated a finite volume and the entire space is subdivided into cells, each one containing one encoded information [12]. The energy necessary to transmit an information is proportional to the distance of the representing point with respect to the origin. An efficient coding, which minimizes the energy, organizes the volumes associated with the different information in the closest possible packing of similar cells around the origin [13].

Dense packings of equal cells in high dimensions have also applications in the study of analogue–digital converters. In this case, the space of the continuous analogue variables is quantized in a system of cells and the volume inside each cell is associated with one digital information. The quantization error is associated with the extension of the interface between the cells and with the distance between the centre of a cell and its vertices [13].

High-dimensional partition of space has also application in neural networks and complex system dynamics [14–18]. Some relevant properties (such as storage capacity in neural networks and the slow-ageing dynamics of glasses) are associated with the subdivision of the phase-space in the basin of attraction around the stored information or the minima of the energy.

2D and 4D froths (and their duals: triangulations and simplicial decompositions) have relevance in quantum gravity [19–24]. Here the continuous space is divided into cells and the functional integration over all equivalence classes of metrics is replaced with a summation over all the triangulations of the given manifold.

Despite the broad variety of systems which are topologically froths and the large number of studies devoted to them in the literature, very little is known about the structure of froths in dimensions larger than $D = 3$. Froths are disordered cellular structures where the cells are highly correlated. These correlations essentially come from the space-filling condition which locally constrains the cells to pack without leaving any empty space, and globally constrains the froth to tile a manifold with a given curvature. In this paper we study how these local and global conditions determinate the average topological properties of the froth structure and we construct froths in any dimension by cell-division and cell-coalescence transformations. The aim of this paper is to investigate the average structural properties of classes of homogeneous partitions of high-dimensional Euclidean spaces and to give analytical instruments and methodologies for the the investigation of the topological structure of froths in spaces of arbitrary dimensions and curvature.

The plan of the paper is as follows. In section 2, the hierarchical organization of topologically stable divisions of space in cells is studied. In section 3, we discuss a way to generate or modify D-dimensional froths by cell-division and cell-coalescence transformations. In section 4, the fixed points of such transformations are studied and the properties of the associated structures are illustrated. The construction of homogeneous partitions and the comparison of their properties with known structures, is considered in section 5.

2. Hierarchy in the cellular structure

A froth in an arbitrary dimension D is a cellular structure where the incidence numbers (i.e. the average number of elements which are incident on a given lower-dimensional element) are fixed by the stability condition at the minimal value. The cells of a froth in dimension D are D-dimensional irregular polytopes packed together to fill space. The boundaries of these cells are made with $(D-1)$-dimensional polytopes which are bounded by $(D-2)$-dimensional polytopes and so on up to the 0D elements which are the vertices. For example, a 3D froth is made with 3D polyhedra (the cells) which are bounded by 2D polygons (the faces) which are bounded by 1D elements (the edges) finally bounded by 0D elements (the vertices). A characterization of this D-dimensional structure can be given in terms of the numbers of D-dimensional polytopes which are making the froth, in terms of the average numbers of $(D-1)$-dimensional polytopes making the boundary of a given cell and so on counting the number of polytopes making the boundary of the boundary, etc.

The boundary of any k-dimensional polytope of the froth is also a froth in a $(k-1)$-dimensional elliptic space. The D-dimensional froth is therefore a graded topological set: it contains D-polytopes, the cells, which are tiling a space which can be Euclidean, elliptic or hyperbolic. The boundary of each cell is an elliptic $D-1$ surface which is tiled by a $(D-1)$-dimensional froth, whose cells are $(D-1)$-polytopes which are the interfaces bounding the original cell and separating it from its topological neighbours. Each interface of the $(D-1)$-froth is an elliptic $(D-2)$-froth of $(D-2)$-polytopes, which are separating the cells from their neighbours. The graded topological set terminates with edges (segments or convex 1-polytopes), bounded by two vertices (or 0-polytopes).

Let us denote with C_k the number of k-dimensional cells in the froth (C_0 number of vertices, C_1 number of edges, C_2 number of faces, C_3 number of polyhedra... C_D number of D-dimensional cells). Let us denote with $\langle n_{i,j}\rangle$ (for $i \leqslant j$) the average number of i-dimensional cells which are surrounding and making the boundary of a j-dimensional cell ($\langle n_{0,1}\rangle = 2$ number of vertices surrounding an edge, $\langle n_{1,2}\rangle$ number of edges per faces, etc).

The average froth structure is characterized by the numbers of polytopes C_i and by the valences $\langle n_{i,j}\rangle$ (with $0 \leqslant i < j \leqslant D$), which are therefore the variables of the problem. The total number of these variables is $\frac{1}{2}(D+1)(D+2)$, but they are related by the Euler equations and constrained by the stability condition. In particular, the numbers of elements in the froth C_i are related through the Euler relation:

$$\sum_{i=0}^{D}(-1)^i C_i = \chi_D \tag{1}$$

where χ_D is the Euler–Poincaré characteristic associated with space curvature. For D even, opposite signs of χ_D correspond to spaces with opposite curvature: $\chi_D > 0$ corresponds to an elliptic D-dimensional space and $\chi_D < 0$ corresponds to a hyperbolic space. On the other hand, for D odd, this relation between the sign of χ_D and the space curvature no longer holds.

A Euler relation is also satisfied for each froth of the graded topological set. That gives a set of relations for the quantities $\langle n_{i,j}\rangle$

$$\sum_{i=0}^{J-1}(-1)^i \langle n_{i,J}\rangle = 1-(-1)^J \qquad \text{with } J = 1, 2, \ldots, D. \tag{2}$$

Here the factor $1-(-1)^J = \chi_{J-1}^{(\text{elliptic})}$ is the Euler–Poincaré characteristic for the surface of a J-dimensional sphere (which is a $(J-1)$-dimensional elliptic space).

The numbers C_i and the averages $\langle n_{i,j}\rangle$ are related by the stability condition

$$\binom{D+1-i}{j-i} C_i = \langle n_{i,j}\rangle C_j \qquad \text{with } i \leqslant j \leqslant D \tag{3}$$

where $n_{i,i} = 1$. The binomial coefficient in the left-hand side of equation (3) is an incidence number (number of j-dimensional polytopes incident on an i-dimensional polytope) which is fixed at the minimum value by the stability condition (there are $(D+1)$ edges and $(\frac{D+1}{2})$ faces incident on each vertex, D faces incident on each edge, etc).

By contrast, the coordination numbers $\langle n_{i,j}\rangle$ with $i < j$ are variables (except for $n_{0,1}$: every edge is bounded by two vertices and therefore $n_{0,1} = 2$). These variables are not all independent and their range of variability is severely restricted by relations (1)–(3) (for example, in a 2D-infinite Euclidean froth we have $\langle n_{1,2}\rangle = \langle n_{0,2}\rangle = 6$ in consequence of the Euler relation).

One can show that [25] a complete topological characterization of the average structure of a D-froth is given by a set of $\frac{D}{2} - 1$ (or $\frac{D-1}{2}$) for D even (or odd), independent variables: the even 'valences' $X_{2l} = \langle n_{2l-1,2l}\rangle$ with $2l < D$. These valences are the *average coordination numbers* (average number of neighbours) of the $2l$-dimensional polytopes in the froth. These are *free variables*. In contrast, the coordination numbers for the odd-dimensional polytopes (the odd valences) are given in terms of the even valences by the relations

$$\frac{X_1X_2\ldots X_J}{(J)!} - \frac{X_2\ldots X_J}{(J-1)!} + \cdots - (-1)^J \frac{X_{J-1}X_J}{2} + (-1)^J X_J = 1 - (-1)^J \qquad \text{for } J = 1, 2, \ldots D. \tag{4}$$

(This is obtained from equation (2) associated with equation (3) and by using the definition $X_k = \langle n_{k-1,k}\rangle$.) For $J = 2l+1$ odd, the left-hand term in equation (4) is equal to 2 and equation (4) fixes the value of the odd valences X_{2l+1} in terms of the even ones. When $J = 2l$ even, the left-hand term is zero and therefore the even valences X_{2l} are free variables.

In k dimensions the polytope with minimal coordination number is a simplex with $k+1$ neighbours. Therefore, the valences must stay in the range $k+1 \leqslant X_k < \infty$. The average structure of a D-dimensional froth is characterized by a sequence of even free valences $\{X_2, X_4, X_6\ldots\}$. For any given sequence of even valences $\{X_{2l}\}$, the odd ones can be calculated by using equation (4). But only sequences which generate odd valences with $X_{2l+1} \geqslant 2l+2$ are admissible. This condition strongly constrains the accessible values of the even valences.

The Euler relation (1), associated with equations (4) and (3), gives an additional relation between valences

$$\left(\frac{X_1X_2\ldots X_D}{(D+1)!} - \frac{X_2\ldots X_D}{(D)!} + \cdots - (-1)^D \frac{X_D}{2} + (-1)^D\right) C_D = \chi_D. \tag{5}$$

In even-dimensional spaces, the sign of the Euler–Poincaré characteristic χ_D is associated with the space curvature. The sign of the term inside the brackets in the left-hand side of equation (5) is the same as that of χ_D (because $C_D > 0$). Therefore, any two regions of the valences' space which have different signs to the bracket term (i.e. to χ_D) correspond to two froths on manifolds of opposite Gaussian curvature. For example, in the 2D case, where the average structure of the froths is described by X_2, equation (5) gives

$$(6 - X_2)\frac{C_2}{2} = \chi_2 \tag{6}$$

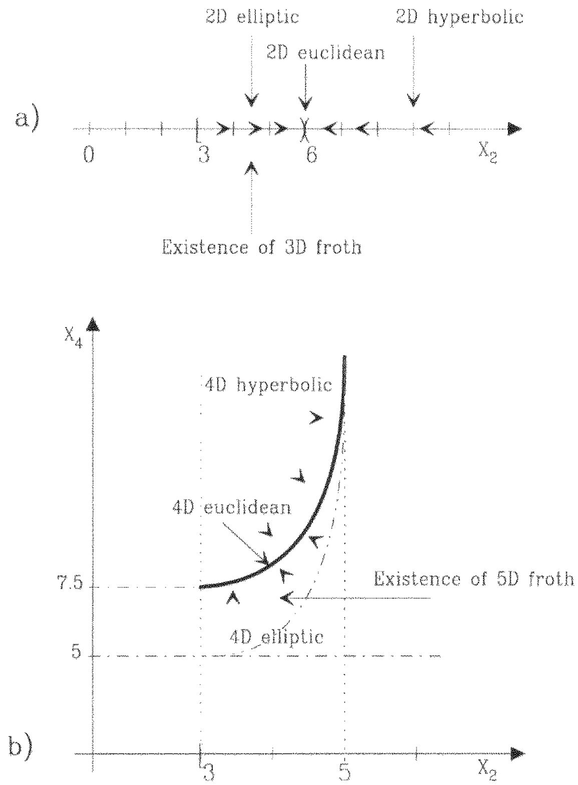

Figure 1. Different regions in the parameter space $\{X_{2l}\}$ correspond to froths which are tiling spaces of different curvatures. (a) 2D froths with $X_2 < 6$ tile elliptic surfaces, froths with $X_2 > 6$ tile hyperbolic surfaces, and $X_2 = 6$ corresponds to froths tiling the Euclidean plane. (b) The hyperbolic, Euclidean and elliptic tilings correspond to three regions of the $\{X_2, X_4\}$ parameter space. Cell-division transformations modify the properties of curved tilings towards the Euclidean ones (arrows).

which indicates that 2D froths with $X_2 < 6$ are tiling elliptic surfaces, whereas froths with $X_2 > 6$ are tiling hyperbolic surfaces (see figure 1(a)). In 4D equation (5) gives

$$X_4 = \left(1 - \frac{\chi_4}{C_4}\right) 5 \frac{6 - X_2}{5 - X_2} \tag{7}$$

(where we used equation (4) to express X_3 as a function of X_2). Equation (7) indicates that the region in the parameter space $\{X_2, X_4\}$ below the line $X_4 = 5\frac{6-X_2}{5-X_2}$ is associated with 4D froths which are tiling elliptic manifolds ($\chi_4 > 0$), whereas the region above this line correspond to froths tiling hyperbolic manifolds ($\chi_4 < 0$) (see figure 1(b)).

Note that equation (4) is a constraint on the sequence $\{X_{2l}\}$ due to local conditions (it concerns the average properties of a $(2l+1)$-dimensional polytope in terms of the properties of the lower-dimensional elements that are making it), and equation (5) is a constraint on $\{X_{2l}\}$ due to the global curvature of the manifold that the froth is tiling.

Note also that relation (7) has a singularity in $X_2 = 5$. This is associated with the existence of polytope $\{5, 3, \ldots\}$ (Schläfly symbols [26]) up to $D = 4$ only, as explained in appendix A.

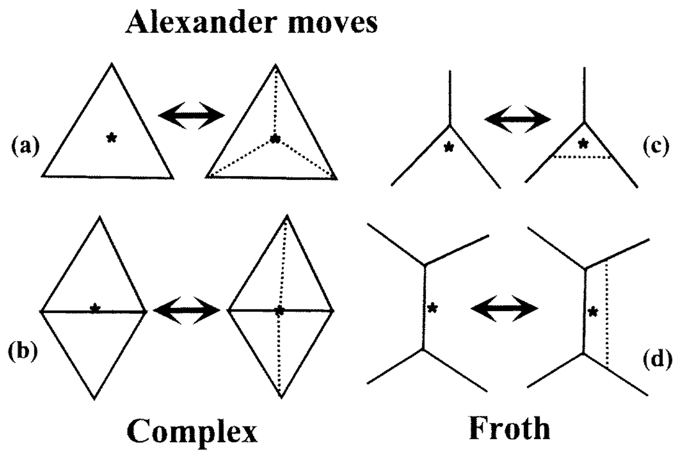

Figure 2. Given two different partitions of the D-dimensional space in D-simplexes one can be transformed into the other by a finite sequence of two local transformations called 'Alexander moves' ((a) and (b) for the 2D case). In the dual froths these transformations correspond to two special cell-division transformations ((c) and (d) for the 2D case).

3. Cell-division and cell-coalescence transformations

In this section we build D-dimensional froths by using cell-division transformation and its inverse (cell-coalescence). This is a local transformation that changes the structure of the froth but leaves unchanged the global topological properties (the curvature of the manifold tiled by the froth or, equivalently, the parameter χ_D). By using cell division and coalescence it is therefore possible to generate different froths which are tiling topologically identical manifolds.

In the literature analogous transformations have been studied for the dual problem of triangulations and simplicial decomposition. In particular, it is known that, given two different partitions of the D-dimensional space in D-simplexes (where a zero-simplex is a point, one-simplex an edge, two-simplex a triangle, three-simplex a tetrahedron, etc), one can be transformed into the other by a finite sequence of two local transformations called 'Alexander moves' [27]. The first move is the addition of a vertex inside a simplex dividing it into $D+1$ simplexes with the same boundary as the original simplex. The second move consists of adding a vertex on an edge of a simplex and connecting it with the vertices of the incident simplexes. For the 2D case the two Alexander moves are the insertion of a new vertex inside a triangle and the insertion of a new vertex on an existing edge. They are shown in figures 2(a) and (b). In the dual froth these moves correspond to a cell division which inserts a triangle near to an existing vertex and to a cell-division which inserts a square near to an existing edge (figures 2(c) and (d)). In 3D one can easily see, by following the same procedure illustrated for 2D, that the two Alexander moves can be obtained in the dual space of the froth by applying cell-division transformations. In the general case, one can see that the first Alexander move can always be done in the dual froth by dividing a cell in the proximity of a vertex, inserting in this way a new polytope with $D+1$ neighbours (a simplex). The second move can be done by dividing a cell in the proximity of an existing $(D-1)$-dimensional interface between two cells. In this case, the kind of polytope inserted depends on the local configuration.

We have therefore shown that the two Alexander moves are reduced in froths to two special kinds of cell-division transformations. Consequently, *the entire set of all the possible froths tiling a given manifold can be generated by cell-division and its inverse (cell-coalescence) transformations*.

Now we investigate how the average properties of the structure are modified by these transformations. First consider the cell-division transformation in the 2D case. The cut of a cell corresponds to insert into the system one additional face, three edges and two vertices. Therefore, one has the transformations $C_2 \to C_2 + 1$, $C_1 \to C_1 + 3$ and $C_0 \to C_0 + 2$. One can verify that the Euler–Poincaré characteristic rests unchanged (indeed, $\chi_D = C_0 - C_1 + C_2$). However, the average coordination number (which is given by $X_2 = \frac{2C_1}{C_2}$, see equation (3)) is modified

$$X_2' = X_2 \mp \frac{1}{C_2 \pm 1}(X_2 - 6) \tag{8}$$

with the upper sign corresponding to a cell-division transformation and the lower sign to its inverse (coalescence).

Now consider the 3D case. Cell division consists of inserting inside a cell a new face which can have, in general, c_1 edges and $c_0 (= c_1)$ vertices. This cut corresponds to the transformation $C_3 \to C_3 + 1$, $C_2 \to C_2 + 1 + c_1$, $C_1 \to C_1 + c_1 + c_0$ and $C_0 \to C_0 + c_0$. Equation (3) gives $X_3 = \frac{2C_2}{C_3}$, and therefore we obtain that the average coordination number transforms as

$$X_3' = X_3 \mp \frac{1}{C_3 \pm 1}(X_3 - 2(c_1 + 1)). \tag{9}$$

In the D-dimensional case, the cut of a cell corresponds to introducing a $(D-1)$-dimensional interface which is, in general, made of c_0 vertices, c_1 edges, c_2 faces, c_3 3D cells... c_{D-2} $(D-2)$-dimensional polytopes. Consequently, the division of a D-dimensional cell (or the coalescence between two cells) of the D-dimensional froth corresponds to the transformation

$$\begin{aligned}
&C_0 \to C_0 \pm c_0 \\
&C_1 \to C_1 \pm c_1 \\
&C_2 \to C_2 \pm c_2 \pm c_1 \\
&\vdots \\
&C_k \to C_k \pm c_k \pm c_{k-1} \\
&\vdots \\
&C_{D-1} \to C_{D-1} \pm 1 \pm c_{D-2} \\
&C_D \to C_D \pm 1
\end{aligned} \tag{10}$$

where the upper sign $(+)$ corresponds to a cell-division transformation and the lower sign $(-)$ to its inverse (coalescence). By substituting into equation (1) one can verify that the global curvature (χ_D) is an invariant quantity under the transformation (10). Note that expression (10) takes the canonical form $C_k \to C_k \pm c_k \pm c_{k-1}$ for all k if one imposes $c_{-1} = 0$, $c_{D-1} = 1$ and $c_D = 0$.

From equation (3) one has the identity $X_k C_k = (D + 2 - k)C_{k-1}$ (where we used the definition $X_k = \langle n_{k-1,k} \rangle$). By substituting in this expression the transformation (10) we get

$$X_k' = X_k \mp \frac{1}{C_k \pm c_k \pm c_{k-1}}(X_k(c_k + c_{k-1}) - (D + 2 - k)(c_{k-1} + c_{k-2})) \tag{11}$$

(upper sign, cell division; lower sign, cell coalescence).

We recall that through cell-division/coalescence transformations it is possible to generate the full class of froths tiling topologically identical manifolds. The modification of the average structural properties associated with these geometrical transformation are algebraically given by equation (11). By using this expression it is therefore possible to *find the average topological properties of all the froths on a given manifold*.

4. Fixed points

When cell-division or coalescence transformations are performed on a 2D froth with average coordination $X_2 = X_2^* = 6$, they leave the local average structural properties unchanged (i.e. $X_2' = X_2 = X_2^* = 6$, see equation (8)). This is a fixed point in the transformation (10) and corresponds to 2D Euclidean froths. Moreover, one can see that the average structural properties of froths which are tiling elliptic surfaces (i.e. where $X_2 < 6$) are modified toward the Euclidean structure ($X_2 < X_2' < 6$) by the application of the cell-division transformation. Analogously, hyperbolic froths ($X_2 > 6$) are also modified towards the Euclidean structure ($6 < X_2' < X_2$) (see figure 1(*a*)). (Note that the global curvature remains always unchanged. Indeed, χ_D is invariant under the transformation (10).)

In the general case, one can immediately see that transformation (11) has the fixed point

$$X_k^* = (D + 2 - k)\frac{c_{k-1} + c_{k-2}}{c_k + c_{k-1}} \tag{12}$$

which is the structure that is invariant under cell-division/coalescence transformations ($\{X_k^{*\prime}\} = \{X_k^*\}$).

A froth is a graded set. Therefore the $(D-1)$-dimensional interface that is introduced into the system to cut a cell is a $(D-2)$-dimensional elliptic froth with c_0 vertices, c_1 edges $\ldots c_{D-2}$ $(D-2)$-dimensional cells. All the relations written above, and in particular equations (2) and (3), can be applied to this $(D-2)$-dimensional elliptic froth. One has, $c_k x_k = (D-k)c_{k-1}$ and $c_{k-1}x_{k-1} = (D+1-k)c_{k-2}$, with x_k and x_{k-1} the average coordination numbers of the k and $(k-1)$-dimensional polytopes which are making the $(D-1)$-dimensional interface. By substituting into equation (12), one gets

$$X_k^* = \frac{(D+2-k)(D+1-k+x_{k-1})}{(D+1-k)(D-k+x_k)}x_k. \tag{13}$$

The fixed point configuration is therefore determined by a set of variables $\{x_k\}$ with $k < D$ which are the average coordinations of the $(D-1)$-dimensional polytope that is inserted or removed during the cell-division or coalescence transformation. For example, in $D = 3$, relation (13) gives

$$X_3^* = 2x_2 + 2 \tag{14}$$

with x_2 the number of edges of the face that is inserted (or removed) to divide a cell (or effect coalescence between two cells).

The minimum number of edges per cell is three. Therefore from equation (14) it follows that fixed-point structures are possible only in the region of the parameter space with $X_3 \geqslant 8$ (see figure 1(*a*)). Any structure with $X_3 < 8$ is transformed towards the fixed-point region ($X_3 \geqslant 8$) by applying cell-division transformations.

In 4D equation (13) gives

$$\begin{aligned} X_2^* &= \frac{20x_2}{3(2+x_2)} \\ X_4^* &= 2(x_3+1) = 2\left(\frac{12}{6-x_2}+1\right). \end{aligned} \tag{15}$$

We can express the parameter x_2 in equation (15) in terms of X_2^* obtaining $X_4^* = 5\frac{6-X_2^*}{5-X_2^*}$, which is the condition on the even valences that identifies the Euclidean region in 4D froths (see equation (7)). *The fixed-point structures are Euclidean*. Since $x_3 \geqslant 4$, it follows that $X_4^* \geqslant 10$, which implies that structures generated by cell division can only access a part of the Euclidean region in the phase-space.

We can in general prove that the fixed point $\{X_k^*\}$ given by equation (13), is the average structure of a D-dimensional froth which is tiling a manifold with $\chi_D = 0$. Indeed, let us substitute into equation (5) the fixed-point configuration ($\{X_k^*\}$) and apply the cell-division transformation. By definition the sequence $\{X_k^*\}$ does not change, whereas the total number of cells increases of a unity ($C_d \to C_D + 1$). To satisfy equation (5) before and after this transformation one must have $\chi_D = 0$. Which proves the theorem.

By rewriting equation (11) in the form

$$X_k' = X_k \mp \frac{(c_k + c_{k-1})}{C_k \pm c_k \pm c_{k-1}}(X_k - X_k^*) \tag{16}$$

it is easy to see that the fixed points are stable under cell-division transformations (upper sign in equation (16)) which insert identical polytopes as interface. Indeed, from equation (16), if $X_k > X_k^*$ then $X_k > X_k' > X_k^*$ and vice versa.

In the space of the configuration $\{X_2, X_4, \ldots, X_{2l}, \ldots\}$, when D is even, the Euclidean region is a surface given by equation (5) (with $\chi_D = 0$). The fixed-point configurations are a subset of this surface. Froths outside the fixed-point configuration are always transformed toward this subset by applying cell-division transformations. When D is odd, equation (5) is not a constraint and the fixed points are associated with a sub-volume of the whole accessible parameter space.

5. Construction of Euclidean froths

In this paragraph we study froths generated by cell-division transformations. We therefore study the class of structures $\{X_k^*\}$ given by equations (11) and (13). The full class of these froths is obtained by varying in equation (13) the parameters $\{x_k\}$ in the allowed range (i.e. $k + 1 \leqslant x_k < \infty$, which satisfy the conditions (4) for k odd and the relation (5) with $\chi_D < 0$). Here, we study only some particular cases.

Let us first note that, from equation (13), the average number of neighbours of the fixed-point structures is given in terms of the coordination of the inserted interface (x_{D-1}) by

$$X_D^* = 2x_{D-1} + 2. \tag{17}$$

In 2D an edge is inserted or removed from a face. The ‘coordination’ of an edge is its number of vertices: $x_{D-1} = x_1 = 2$. Therefore $X_2^* = 6$, as it should in Euclidean space. In 3D a face is inserted in, or removed from a cell. The coordination of this face (x_2) is its number of edges and in principle it can be any number between 3 and ∞. But a face with a large number of edges can be inserted only in a cell with a large number of neighbours and it can be removed only if it exists in the froth. Therefore, only some values of x_2 are admissible. One can easily see that a triangle ($x_2 = 3$) can always be inserted in the proximity of a vertex. Analogously, a square ($x_2 = 4$) can also be always inserted in the proximity of an edge. From (17) it follows therefore that 3D Euclidean froths with $8 \leqslant X_3^* \leqslant 10$ can always be generated. But, in general, it should be possible to insert faces with higher values of x_2. To have an estimation for the ‘typical’ value for the number of edges of the inserted face let us make a cut of the whole 3D froth with a plane. The result is a 2D Euclidean froth where each single face is the result of a cut on a 3D cell. This 2D froth is therefore a representative set of faces produced by random cuts of 3D cells. The average number of edges for this set of faces is $x_2 = X_2^* = 6$. Therefore, from (17), a 'typical' froth generated by cell division is expected to have a fixed-point coordination around $X_3^* = 14$ [28]. Cells in biological tissues appear in various polyhedral shapes with a number of faces distributed in a narrow range around 14 [29]. A widely studied 3D froth made with identical cells is the 'Kelvin froth': its cells are space-filling truncated octahedra with $X_3 = 14$ [30, 31]. Coordinations between 15.53 and 14 are found in Voronoï partitions of space [32], where the higher value corresponds to a Voronoï partition from random points [35], whereas the lower value corresponds to more compact and homogeneous packings. Smaller values in the range $13.333 \leqslant X_3^* \leqslant 13.5$ characterize an interesting class of natural structures (Frank–Kasper phases [33]) which partition ordinary space with cells with pentagonal and hexagonal faces only. Soap froth has typically $X_3 \simeq 13.7$ [34].

In a D-dimensional froth a $(D-1)$-dimensional interface with coordination x_{D-1} is inserted or removed by cell-division or coalescence transformations. As pointed out above, simplexes with coordinations $x_{D-1} = D$ can always be inserted in the proximity of an existing vertex. This is the minimum possible value for x_{D-1}, substituted into equation (17) it sets the minimum value of the average number of neighbours in a D-dimensional Euclidean fixed-point structure at the *minimal coordination*

$$X_D^* = 2D + 2. \tag{18}$$

The argument for the 'typical' cut that we used in 3D can be directly extended to any dimension. Indeed, one of the properties of froths is that a cut with a hyperplane of a D-dimensional froth generates a $(D-1)$-dimensional Euclidean froth. For instance, a cut of a 4D froth gives a 3D Euclidean froth. We can assume, that this froth has the 'typical' coordination $X_3^* = 14$ found above. Inserting into equation (17) one gets $X_4^* = 30$. The same arguments, extended to any dimension, give

$$X_D^* = 2^{(D+1)} - 2 \tag{19}$$

for the average number of neighbours per cell in the 'typical' D-dimensional froth.

What makes equation (13) powerful is the fact that, not only the average number of neighbours, but *all the average properties of the fixed-point structures can be deduced in terms of the properties of the inserted interfaces*.

5.1. Minimally coordinated froths

Let us first construct the Euclidean froth with minimum coordination numbers. It is the fixed-point structure associated with a cell-division transformation which inserts interfaces with minimal coordinations. These interfaces are $(D-1)$-dimensional simplexes inserted in the proximity of a vertex. They have $x_k = k+1$ (for $k < D$). By substituting into equation (13) one obtains

$$X_k^* = \frac{D+2-k}{D+1-k}(k+1). \tag{20}$$

(Note that $X_D^* = 2D+2$, as discussed above.) These are the average structural properties of a froth which is tiling a manifold with $\chi_D = 0$ which is homologous to the Euclidean space. It is the known Euclidean froth with minimal coordination numbers. Starting from any given D-dimensional froth one can always transform it into this minimally coordinated one by applying an infinite number of cell divisions near existing vertices. The resulting structure is expected to have cells with very different topological properties. Indeed, for each cell-division transformation a new cell with $D+1$ neighbours is inserted and one neighbour is added to the $D+1$ cells around the inserted simplex, distributing therefore the coordinations inhomogeneously between cells.

5.2. Homogeneous partitions

A D-dimensional froth has $D+1$ edges incident on each vertex. In an ideal homogeneous partition of space these edges are equally separated in angle. That corresponds to an angle $\theta^{\rm ideal} = \cos^{-1}(-1/D)$ between each couple of edges [25]. In a froth, edges must close in rings which are bounding 2D faces. It is easy to see that with the angle $\theta^{\rm ideal}$, flat rings close with an average number of edges equal to

$$X_2^{\rm ideal} = \frac{2\pi}{\pi - \cos^{-1}(-\frac{1}{D})}. \tag{21}$$

This number is 6 in 2D, 5.104 in 3D, 4.767 for $D = 4$, 4.588 for $D = 5$ and tends to 4 when $D \to \infty$. Note that $X_2^{\rm ideal}$ is irrational for $D > 2$. In the Euclidean space, the 'ideal' structure cannot be obtained by any ordered lattice structure. This is an example of geometrical frustration. But disordered or non-periodic structures can approximate $X_2^{\rm ideal}$ with arbitrary precision avoiding in this way the frustration.

The average number of edges per cell $X_2^{\rm ideal}$ (equation (21)) is the only quantity of the ideal structure that can be calculated by using these geometrical arguments. All the other coordinations are unknown, but we can construct fixed-point structures that approximate this ideal froth in the ring coordination X_2. For these structures we can calculate the whole set of coordinations, and therefore we can infer information about the coordinations of the ideal one.

We expect that structures which uniformly partition space must have $X_2 \simeq X_2^{\rm ideal}$. By cell-division transformation it is possible to generate froths that approximate the ideal structures by inserting interfaces with x_2 close to the ideal value $X_2^{\rm ideal}$.

For $D = 3$, $X_2^{\rm ideal} = 5.104$ which corresponds to $X_3^{\rm ideal} = 13.392$. We can generate homogeneous partitions by inserting pentagons $x_2 = 5$ or hexagons $x_2 = 6$ obtaining (from equation (17)) $12 \leqslant X_3^* \leqslant 14$, which is in the right range.

In general, since we are looking for homogeneity, it is logical to insert as interfaces regular polytopes with x_2 close to the ideal value $X_2^{\rm ideal}$. These polytopes can only be hypercubes $\{4, 3, \ldots,\}$ (which have $x_2 = 4$) and the polytope $\{5, 3, \ldots,\}$ (with $x_2 = 5$), but this second polytope exists only up to $D = 4$ ([26] and appendix A).

5.2.1. Cell division with $\{5,3,\ldots\}$ *(the (a) plots in figure 3).* Cell-division operations which insert the polytope $\{5, 3, \ldots\}$ can therefore generate Euclidean fixed-point structures up to $D = 5$. In $D = 3$ this corresponds to cell divisions which insert pentagonal faces, that generate a fixed-point structure with $X_2^* = 5$ and $X_3^* = 12$. In 4D, the fixed-point structure obtained by inserting dodecahedra ($\{5,3\}$, $x_2 = 5$, $x_3 = 12$) has $X_2^* = 100/21 = 4.7619\ldots$, a value that is very close to the ideal one. The 4D cells of this froth have $X_4^* = 26$ neighbours on average. For $D = 5$, by dividing cells with the polytope $\{5,3,3\}$ ($x_2 = 5$, $x_3 = 12$, $x_4 = 120$) we obtain a fixed-point structure with $X_2^* = 75/16 = 4.68\ldots$ (see equation (13)), which is larger than the ideal value. The average number of neighbours is in this case $X_5^* = 242$.

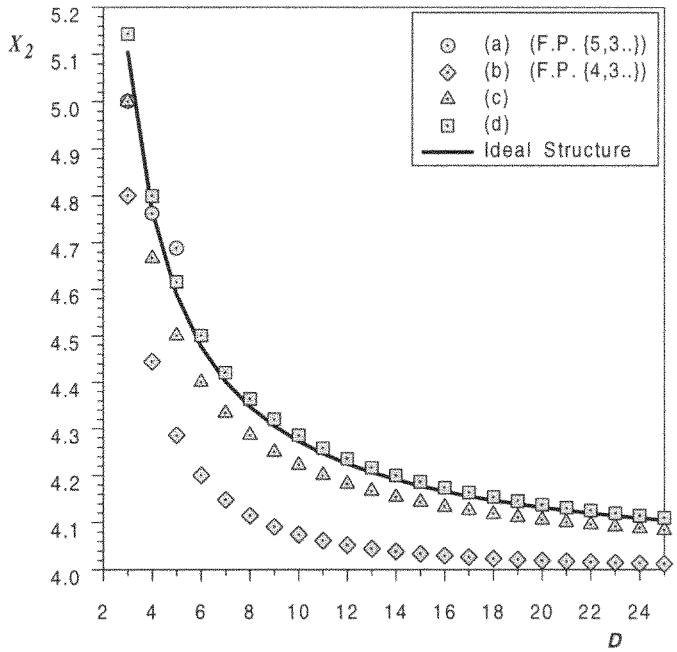

Figure 3. The average ring connectivity X_2, for homogeneous partitions of spaces of dimension D, obtained from a geometrical approach [25] (full curve) is compared with those of the fixed-point structures (symbols).

5.2.2. Cell division with $\{4,3,\ldots\}$ *(the (b) plots in figure 3).* The average coordinations of the fixed-point froths generated by inserting hypercubes $\{4, 3, \ldots\}$ are given by imposing $x_k = 2k$ into equation (22))

$$X_k^* = \frac{(D+2-k)(D-1+k)}{(D+1-k)(D+k)} 2k. \tag{22}$$

In this structure the D-dimensional cells have $X_D^* = 4D - 2$ neighbours on average. The average ring coordination is $X_2^* = 4D(D+1)[(D-1)(D+2)]^{-1}$ which correctly tends to 4 when $D \to \infty$, but it is consistently lower than X_2^{ideal} for $D > 2$. This is presumably a rather inhomogeneous structure.

5.2.3. Homogeneous cell division (the (c) plots in figure 3). To maximize homogeneity one can construct a structure by inserting new interfaces with the same topological properties as the existing structure. We expect a resulting structure that evolves towards a self-uniform homogeneous partition. Let us therefore perform cell-division transformations by inserting interfaces with $x_k = X_k^*$ (with $k = 2, \ldots, D-1$). By substituting into equation (13), one obtains a recursive equation with the following solution

$$X_k^* = \left(1 + \frac{D}{D+1-k}\right) k. \tag{23}$$

Here $X_d = D(D+1)$ and $X_2 = 2\frac{2D-1}{D-1}$, which asymptotically tends to 4 and is much closer to the ideal value than the one of structure (b).

Table 1. Average ring coordination (X_2) and cell coordination (X_D) for some Euclidean fixed-point structures (FP) generated by cell division (see text).

		FP $\{5,3,\ldots\}$ (a)		FP $\{4,3,\ldots\}$ (b)		FP (c)		FP (d)	
D	Ideal X_2^{ideal}	X_2^* $5\frac{D(D+1)}{(D-1)(D+3)}$	X_D^* $4\frac{D(D+1)}{(D-1)(D+2)}$	X_2^* $4D-2$	X_D^*	X_2^* $2\frac{2D-1}{D-1}$	X_D^* $D(D+1)$	X_2^* $12\frac{D}{3D-2}$	X_D^* $2^{D+1}-2$
3	**5.1043**	**5**	**12**	4.8	10	**5**	**12**	**5.1458**	**14**
4	**4.7668**	**4.7619**	**26**	4.4444	14	**4.6667**	**20**	4.8	30
5	**4.5881**	4.6875	242	4.2857	18	**4.5**	**30**	**4.6154**	**62**
6	**4.7728**			4.2	22	**4.4**	**42**	**4.5**	**126**
7	**4.4017**			4.1481	26	**4.3333**	**56**	**4.4211**	**254**
8	**4.3468**			4.1143	30	**4.2857**	**72**	**4.3636**	**510**
9	**4.3052**			4.0909	34	**4.25**	**90**	**4.32**	**1022**
10	**4.2724**			4.0747	38	**4.2222**	**110**	**4.2857**	**2046**

5.2.4. Cell division with 'typical' interfaces (the (d) plots in figure 3). Partitions can be generated by inserting 'typical' interfaces as described before. Here the 'typical' $(D-1)$-dimensional interface has coordinations x_k which are equal to average coordinations of the fixed-point Euclidean structure obtained with this procedure in $D-1$ dimensions. The values X_k^* are then given in term of a recursive equation (with initial condition $X_1^* = 2$). Here are the solutions for $k = 2$ and $k = D$, which have a simple compact form

$$
\begin{aligned}
X_2^* &= 12\frac{D}{3D-2} \\
X_D^* &= 2^{(D+1)} - 2.
\end{aligned}
\tag{24}
$$

Surprisingly the product of these valences from $k = 1$ to D also has a very simple form: $X_1^* X_2^* X_3^* \ldots X_D^* = D!(D+1)!$. Here, the value of X_2 is larger than that of the ideal structure but is extremely close to it.

In table 1 the values of X_2 and X_D are reported, up to $D = 10$, for the whole set of fixed-point structures which have been studied in this section. In figure 3 the value of X_2 for the ideal partition and for the fixed-point structures (a)–(d) are plotted up to $D = 24$.

5.3. Kissing numbers

The average number of neighbours X_D of a D-dimensional cell in a Euclidean froth is an interesting quantity. In sphere packings a corresponding quantity is called the 'kissing number' (KN), that is the number of identical spheres that can be placed around a given sphere being in contact (kissing) with it [13]. To find sphere-packing configurations with high KNs has relevance in the design of efficient codes. It is known that, for packings of identical spheres, the KN is 6 in $D = 2$, 12 in $D = 3$, but exact answers are unknown for dimensions above three except for $D = 8$ (KN = 240) and $D = 24$ (KN = 196 560) where two especially dense lattices E_8 and A_{24} achieve the maximal possible values of KN. In figure 4 are reported the values of the highest known KNs for lattice and non-lattice sphere packings. Two known bounds for KN when $D \gg 1$ are also reported. The KN question concerns finding the best local arrangements of spheres. In high-dimensional spaces, this configuration does not necessarily take the form of a lattice packing. Disordered or quasi-ordered packings are often more suitable to attain high KN. Dimension $D = 9$ is the first where non-lattice packings are known to be superior. Here the Leech lattice Λ_9 has KN = 272 whereas the best bound known is 380 [13].

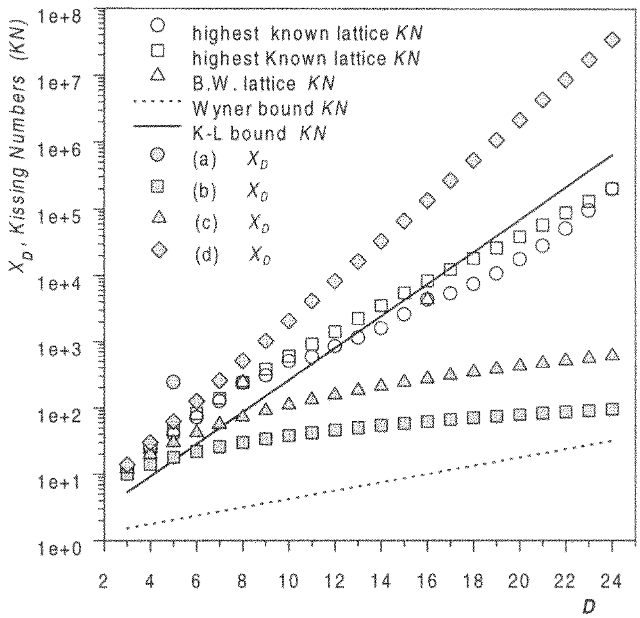

Figure 4. Kissing numbers (KN—maximum number of identical hard spheres that can touch a given sphere in D dimensions) and coordination numbers X_D (average number of cells around a given cell in a D-dimensional froth) are compared for some known sphere packings (open symbols) and for the fixed-point structures (full symbols). Two upper and lower bounds for the KN in high dimensions are also plotted (full and dotted lines).

With any sphere packing one can associate a cellular structure constructed by partitioning the space into convex polytopes each one containing inside it a sphere. Kissing spheres are neighbours. In a dense sphere packing the enveloping polytopes make a space-filling partition of space. The number of neighbours for this system for polytopes is related to the KN and is expected to be bigger than KN, because some non-kissing spheres can be first neighbours in the associated froth. This is for instance the case in $D = 3$ where the configuration with KN $= 12$ corresponds to a close packing of spheres with an associated Wigner–Seitz cell that do not pack in a froth: the incidence numbers are not minimal. This is a *topologically unstable* configuration. Infinitesimal random displacements change the number of topological neighbours from 12 to an average value of 14, but in this case neighbouring spheres will be not all in contact. In general, in close packings, we expect the number of neighbours X_D of the enveloping polytopes to be bigger than, but of the same order of magnitude as, the KNs of the enveloped spheres.

In figure 4 the KNs for some known sphere packings are compared with the coordination numbers obtained from our homogeneous partitions (a)–(d) up to $D = 24$.

5.4. *Voronoï partitions*

The average number of vertices on the boundary of a D-dimensional cell ($\langle n_{0,D}\rangle$) can be exactly calculated for Voronoï partitions generated from random points [35, 36]:

$$\langle n_{0,D}\rangle = \frac{2}{D}\frac{\Gamma(D)}{\Gamma(\frac{1}{2}(D+1))^2}\left[\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}D+1)}{\Gamma(\frac{1}{2}(D+1))}\right]^{D-1}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(D^2+1))}{\Gamma(\frac{1}{2}D^2)}. \tag{25}$$

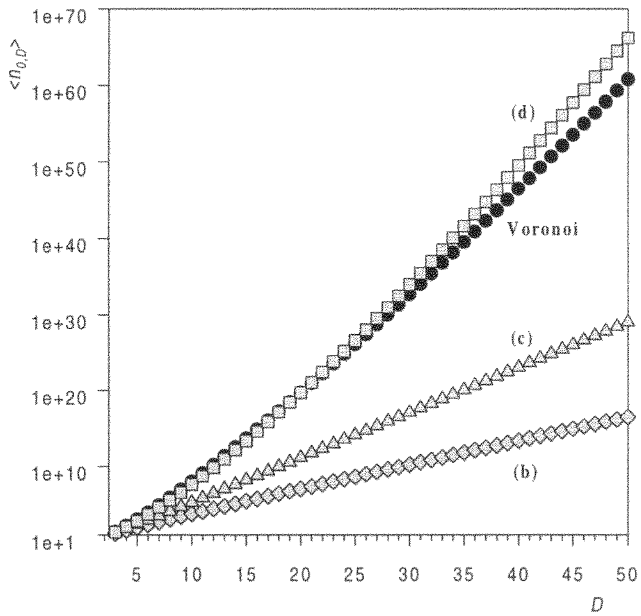

Figure 5. The average number of vertices in a D-dimensional Voronoï cell generated from Poissonian points can be exactly calculated [32] (full circles). Here this number is compared with the one associated with the fixed-point structures (grey symbols).

Asymptotically this quantity scales as $\langle n_{0,D}\rangle \propto D^{D/2-1}$.

The average number of vertices per cell can be expressed in term of the coordinations by

$$\langle n_{0,D}\rangle = \frac{1}{D!} X_1 X_2 \ldots X_{D-1} X_D. \tag{26}$$

(Note that for $D = 3$ equation (25) gives $\langle n_{0,D}\rangle = 96\pi^2/35$, that when substituted into equation (26) leads to $X_3 = 2 + 48\pi^2/35 = 15.53\ldots$.) By substituting the fixed point configuration into (26) we find that the structure (b) has $\langle n_{0,D}\rangle = (D+1)2^{D-1}$, the structure ($c$) gives $\langle n_{0,D}\rangle = (2D)!(D!)^{-2}$, whereas (d) has $\langle n_{0,D}\rangle = (D+1)!$. In figure 5 the behaviours of the average number of vertices per cell in the Voronoï froth and in the three structures (b), (c) and (d) is shown for $3 \leqslant D \leqslant 50$.

6. Conclusions

The topological structure of a D-dimensional cellular system can be characterized, on average, in terms of the coordinations (X_k) of the irregular polytopes which constitute the structure (section 2). Only the coordinations of the even-dimensional polytopes (X_{2l}) are necessary for this characterization: the odd ones are expressed in terms of the even ones by the relation (4). Therefore, the average structure of a D-dimensional froth is characterized by a sequence $\{X_2, X_4, X_6\ldots\}$ of $\frac{D}{2} - 1$ (or $\frac{D-1}{2}$) variables for D even (or odd). These variables are related with the space curvature through equation (5). Regions in the parameter space $\{X_{2l}\}$ corresponding to D-dimensional froths tiling spaces of different curvature are discussed for $D \leqslant 4$ (figure 1 and equations (6) and (7)).

We used cell-division and coalescence transformations to build D-dimensional froths (section 3). We showed that through cell-division/coalescence transformations it is possible to generate the entire class of froths tiling topologically identical manifolds. The dynamical renormalization of the variables X_k under such transformations is found (equation (11)). The existence of classes of structures which are invariant (fixed points) under cell-division/coalescence was pointed out (equations (12) and (13)). We showed that these structures are tiling Euclidean spaces.

Several fixed-point Euclidean structures were constructed in section 5. We discussed the average statistical properties of minimally coordinated Euclidean froths, and for several topologically homogeneous space partitions (equations (20)–(24) and table 1). The topological properties of the most homogeneous cellular partitions were examined and compared with known geometrical results (figure 3).

Finally, the fixed-point Euclidean structures were compared with known high-dimensional structures generated by sphere packings and Voronoï constructions (figures 4 and 5).

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Appendix A. The existence of polytopes $\{5, 3, \ldots\}$ up to $D = 4$

There is a class of regular polytopes with $X_2 = 5$ and minimally connected vertices ($\{5, 3, ...\}$) which exists up to dimension $D = 4$ [26].

They are pentagons in $D = 2$, dodecahedra ($\{5, 3\}$) in $D = 3$ and polytopes $\{5, 3, 3\}$ in $D = 4$. Tessellation of pentagons makes a 2D elliptic froth with $X_2 = 5$ and $C_2 = 12$, which is a dodecahedron $\{5, 3\}$. Tessellation of dodecahedra makes a 3D elliptic froth with $X_2 = 5$, $X_3 = 12$, $C_3 = 120$, which is the $\{5, 3, 3\}$ structure. It turns out that tessellation with $\{5, 3, 3\}$ polytopes will not make any 5D polytope [26]. If existing, such a structure would be a 4D polytope with $X_2 = 5$, $X_3 = 12$, $X_4 = 120$. By substituting these values into equation (7), we get $C_4 = \chi_4$, which implies $\chi_4 > 0$. This hypothetical structure would therefore be an elliptic froth, isomorphic to a sphere, which implies $\chi_4 = 2$. Then, from the previous identity, $C_4 = 2$. The hypothetical $\{5, 3, 3, 3\}$ structure would be an elliptic froth which closes onto itself with two cells only. But two cells are insufficient to make a 5D polytope (the minimum number is 6). It follows therefore that the $\{5, 3, 3, 3\}$ structure does not exist, and nor do the other higher-dimensional tessellations $\{5, 3, 3, 3 \ldots\}$.

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